Real Analysis GCE Name:___________________________

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Exercise 1:

Let X and Y be two metric spaces and f a mapping from X to Y.
(i). Show that f is continuous if and only if for every subset A of X, $f(\overline{A}) \subset \overline{f(A)}$.
(ii). Prove or disprove: assume that f is injective. Then f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.
(iii). Prove or disprove: assume that X is compact. Then f is continuous if and only if for every subset A of X, $f(\overline{A}) = \overline{f(A)}$.

Exercise 2:

Let $K \subset \mathbb{R}$ have finite measure and let $f \in L^{\infty}(\mathbb{R})$. Show that the function F defined by

$$F(x) := \int_K f(x+t)dt$$

is uniformly continuous on $\mathbb{R}$.

Exercise 3:

Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ such that $f_n \to 0$ a.e.

(i) Show that if $\{f_{2n}\}$ is increasing and $\{f_{2n+1}\}$ is decreasing, then

$$\int f_n \to 0.$$

(ii) Prove or disprove: if $\{f_{kn}\}$ is decreasing for every prime number k, then

$$\int f_n \to 0.$$

(Note on notation: e.g., if $k = 2$, then $\{f_{kn}\}$=$\{f_{2n}\}$. Note also that 1 is not prime).